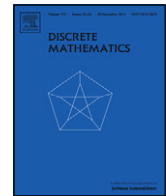


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## Strong edge-coloring for cubic Halin graphs

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## ABSTRACT

A strong edge-coloring of a graph  $G$  is a function that assigns to each edge a color such that two edges within distance two apart must receive different colors. The minimum number of colors used in a strong edge-coloring is the *strong chromatic index* of  $G$ . Lih and Liu (2011) [14] proved that the strong chromatic index of a cubic Halin graph, other than two special graphs, is 6 or 7. It remains an open problem to determine which of such graphs have strong chromatic index 6. Our article is devoted to this open problem. In particular, we disprove a conjecture of Shiu et al. (2006) [18] that the strong chromatic index of a cubic Halin graph with characteristic tree a caterpillar of odd leaves is 6.

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## 1. Introduction

The coloring problem considered in this article has restrictions on edges within distance two apart. The *distance* between two edges  $e$  and  $e'$  in a graph is the minimum  $k$  for which there is a sequence  $e_0, e_1, \dots, e_k$  of distinct edges such that  $e = e_0$ ,  $e' = e_k$ , and  $e_{i-1}$  shares an end vertex with  $e_i$  for  $1 \leq i \leq k$ . A *strong edge-coloring* of a graph is a function that assigns to each edge a color such that any two edges within distance two apart must receive different colors. A *strong  $k$ -edge-coloring* is a strong edge-coloring using at most  $k$  colors. The *strong chromatic index* of a graph  $G$ , denoted by  $\chi'_s(G)$ , is the minimum  $k$  such that  $G$  admits a strong  $k$ -edge-coloring.

Strong edge-coloring was first studied by Fouquet and Jolivet [8,9] for cubic planar graphs. A trivial upper bound is that  $\chi'_s(G) \leq 2\Delta^2 - 2\Delta + 1$  for any graph  $G$  of maximum degree  $\Delta$ . Fouquet and Jolivet [8] established a Brooks type upper bound  $\chi'_s(G) \leq 2\Delta^2 - 2\Delta$ , which is not true only for  $G = C_5$  as pointed out by Shiu and Tam [19]. The following conjecture was posed by Erdős and Nešetřil [5,6] and revised by Faudree et al. [7].

**Conjecture 1.** For any graph  $G$  of maximum degree  $\Delta$ ,

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even;} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{if } \Delta \text{ is odd.} \end{cases}$$

Faudree et al. [7] also asked whether  $\chi'_s(G) \leq 9$  if  $G$  is cubic planar. If this upper bound is proved to be true, it would be best possible. For graphs with maximum degree  $\Delta = 3$ , Conjecture 1 was verified by Andersen [1] and by Horák et al. [12]

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independently. For  $\Delta = 4$ , while [Conjecture 1](#) says that  $\chi'_s(G) \leq 20$ , Horák [11] obtained  $\chi'_s(G) \leq 23$  and Cranston [4] proved  $\chi'_s(G) \leq 22$ . In addition to [Conjecture 1](#), other aspects of strong edge-coloring have also been studied extensively (cf. [1–3,10,13–18,20]).

The main theme of this paper is to study strong edge-coloring for the following planar graphs. A *Halin graph*  $G = T \cup C$  is a plane graph consisting of a plane embedding of a tree  $T$  each of whose interior vertex has degree at least 3, and a cycle  $C$  connecting the leaves (vertices of degree 1) of  $T$  such that  $C$  is the boundary of the exterior face. The tree  $T$  and the cycle  $C$  are called the *characteristic tree* and the *adjoint cycle* of  $G$ , respectively. Strong chromatic index for Halin graphs was first considered by Shiu et al. [18] and then studied in [19,13,14].

A *caterpillar* is a tree whose removal of leaves results in a path called the *spine* of the caterpillar. For  $k \geq 1$ , let  $\mathcal{G}_k$  be the set of all cubic Halin graphs whose characteristic trees are caterpillars with  $k + 2$  leaves. For a graph  $G = T \cup C$  in  $\mathcal{G}_k$ , let  $P: v_1, v_2, \dots, v_k$  be the spine of  $T$  and each  $v_i$  is adjacent to a leaf  $u_i$  for  $1 \leq i \leq k$  with  $v_1$  (resp.  $v_k$ ) adjacent to one more leaf  $u_0 = v_0$  (resp.  $u_{k+1} = v_{k+1}$ ). We draw  $G$  on the plane by putting the path  $v_0Pv_{k+1}$  horizontally in the middle, and the pending edges (leaf edges)  $v_iu_i$ ,  $1 \leq i \leq k$ , by either up or down edges vertically to  $P$ . See [Fig. 1](#) for an example of  $\mathcal{G}_8$ .

From this drawing, we associate  $G$  with a list of positive integers  $(n_1, n_2, \dots, n_r)$ , where  $n_i$  is the number of maximum consecutive up or down edges, starting from the leftmost to the rightmost on  $P$ . We use  $G_{n_1, n_2, \dots, n_r}$  to denote this graph. For instance, the graph in [Fig. 1](#) is  $G_{2,3,3}$ . Notice that  $n_1 + n_2 + \dots + n_r = k$ . For a special case when these pending edges are all in the same direction (up or down), the graph  $G_k$  is called the *necklace* and is denoted by  $Ne_k$  in [18]. Notice that  $G_k$  is the only graph in  $\mathcal{G}_k$  for  $k \leq 3$ .

**Observation 1.**  $G_{n_1, n_2, \dots, n_r} \cong G_{n_r, \dots, n_2, n_1}$ .

**Observation 2.**  $G_{n_1, n_2, \dots, n_r, 1} \cong G_{n_1, n_2, \dots, n_r+1}$ .

It is easy to see that  $\chi'_s(G) \geq 6$  for any  $G \in \mathcal{G}_k$ ,  $k \geq 1$ . Shiu et al. [18] obtained the following results.

•

$$\chi'_s(G_k) = \begin{cases} 9, & k = 2; \\ 8, & k = 4; \\ 7, & k \text{ is even and } k \geq 6; \\ 6, & k \text{ is odd.} \end{cases}$$

- If  $G \in \mathcal{G}_k$  with  $k \geq 4$ , then  $6 \leq \chi'_s(G) \leq 8$ .
- If  $G$  is a cubic Halin graph, then  $6 \leq \chi'_s(G) \leq 9$ .

Moreover, the authors [18] raised the following conjectures.

**Conjecture 2.** If  $G \in \mathcal{G}_k$  with  $k \geq 5$ , then  $\chi'_s(G) \leq 7$ .

**Conjecture 3.** If  $G \in \mathcal{G}_k$  with odd  $k \geq 5$ , then  $\chi'_s(G) = 6$ .

**Conjecture 4.** If  $G = T \cup C$  is a Halin graph, then  $\chi'_s(G) \leq \chi'_s(T) + 4$ .

Faudree et al. [7] proved, for any tree  $T$ , it holds that  $\chi'_s(T) = \max_{uv \in E(T)} (\deg(u) + \deg(v) - 1)$ . [Conjecture 4](#) was confirmed by Lai et al. [13], who proved a stronger result that  $\chi'_s(G) \leq \chi'_s(T) + 3$  for any Halin graph  $G = T \cup C$  other than  $G_2$  and wheels  $W_n$  with  $n \not\equiv 0 \pmod{3}$ , where  $W_n = K_{1,n} \cup C_n$ . Note that  $\chi'_s(W_5) = \chi'_s(K_{1,5}) + 5$ , and  $\chi'_s(G) = \chi'_s(T) + 4$  for  $G = G_2$  or  $G = W_n$  with  $n \not\equiv 0 \pmod{3}$  and  $n \neq 5$ .

[Conjecture 2](#) was confirmed by Lih and Liu [14], who proved a more general result that  $\chi'_s(G) \leq 7$  is true for any cubic Halin graph other than  $G_2$  and  $G_4$ . Hence, strong chromatic index for any cubic Halin graph  $G \neq G_2, G_4$  is either 6 or 7.

It remains open to determine the cubic Halin graphs  $G$  with  $\chi'_s(G) = 6$  (or the ones with  $\chi'_s(G) = 7$ ). Our aim is to investigate this problem. In particular, we establish methods that can be used to study the graphs  $\mathcal{G}_k$ . As a result, we discover counterexamples to [Conjecture 3](#). We prove that for any  $k \geq 7$ , there exists graph  $G \in \mathcal{G}_k$  with  $\chi'_s(G) = 7$ , and for any  $k \neq 2, 4$ , there exists  $G \in \mathcal{G}_k$  (other than necklaces) with  $\chi'_s(G) = 6$ . In [Section 4](#), we determine the value of  $\chi'_s(G)$  for some special families of graphs  $G$  in  $\mathcal{G}_k$ .

## 2. Cubic Halin graphs $G$ with $\chi'_s(G) = 6$

This section gives some cubic Halin graphs with strong chromatic index 6. We begin with the development of several general transformation theorems for Halin graphs.

For a positive integer  $r$ , an  $r$ -tail of a tree  $T$  is a path  $P_r: v_1, v_2, \dots, v_r, v_{r+1}$  in which  $v_1$  is not a leaf but all vertices in  $L_i = \{u \notin P: uv_i \in E(T)\}$  are leaves for  $1 \leq i \leq r$ . For integer  $s < r$ , cutting  $P_s$  from  $T$  means deleting the vertices  $\{v_1, v_2, \dots, v_{s-1}\} \cup_{1 \leq i \leq s} L_i$  from  $T$ , which results in a tree denoted by  $T \ominus P_s$ . Notice that  $v_s$  becomes a leaf adjacent to  $v_{s+1}$  in  $T \ominus P_s$ .

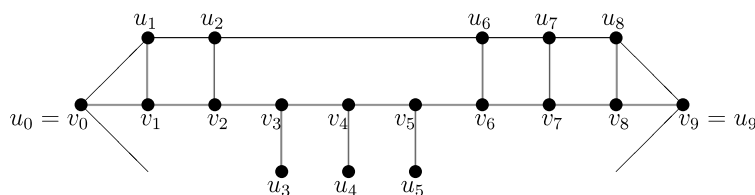


Fig. 1. The graph  $G_{2,3,3}$  in  $\mathcal{G}_8$ .

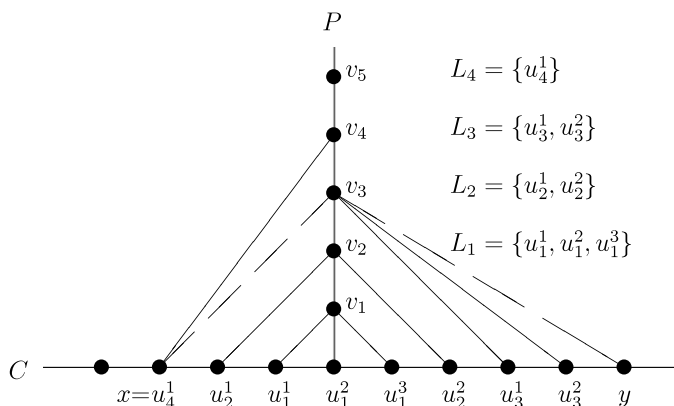


Fig. 2. A cutting 4-tail from  $T$ , resulting in  $G \ominus P_4$  with two new edges,  $v_3x$  and  $v_3y$ , while vertices in  $\{v_1, v_2\} \cup L_1 \cup L_2 \cup L_3$  are all gone.

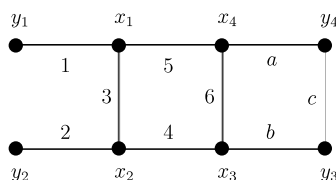


Fig. 3.  $a, b, c$  are forced to be 2, 1, 3, respectively.

Suppose  $P: v_1, v_2, \dots, v_r, v_{r+1}$  is an  $r$ -tail of the characteristic tree  $T$  of a Halin graph  $G = T \cup C$ . For any  $j$  with  $1 \leq j \leq r$ , the vertices in  $\cup_{1 \leq i \leq j} L_i$  form a consecutive portion on the adjoint cycle  $C$ . See Fig. 2 for an example of a 4-tail. For any two vertices in  $\cup_{2 \leq i \leq r} L_i$ , we may regard that they are on the *same* or *different* sides of  $L_1$ . For instance, in Fig. 2,  $u_3^1$  and  $u_3^2$  are on the same side of  $L_1$ , while  $u_2^1$  and  $u_2^2$  are on different sides of  $L_1$ . For  $s < r$ , the tree  $T \ominus P_s$  is the characteristic tree of a new Halin graph, denoted by  $G \ominus P_s$ , whose adjoint cycle is obtained from  $C$  by replacing the segment  $\{x\} \cup_{1 \leq i \leq s} L_i \cup \{y\}$  by the path  $xv_sy$  originally not in  $G$ , where  $x$  (respectively,  $y$ ) is the vertex in  $C$  right before (respectively, after)  $\cup_{1 \leq i \leq s} L_i$ . See the dashed path for  $xv_sy$  in Fig. 2.

We denote a 4-cycle by  $(x_1, x_2, x_3, x_4)$ , which consists of the edges  $x_4x_1$ , and  $x_ix_{i+1}$  for  $i = 1, 2, 3$ .

**Lemma 3.** Suppose  $(x_1, x_2, x_3, x_4)$  is a 4-cycle in a graph  $G$  in which each  $x_i$  is adjacent to a vertex  $y_i$  not in the 4-cycle for  $1 \leq i \leq 4$ . If  $\chi'_s(G) = 6$ , then for every strong 6-edge-coloring  $f$  of  $G$  we have

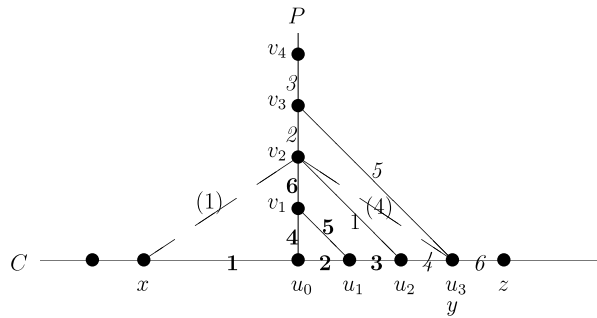
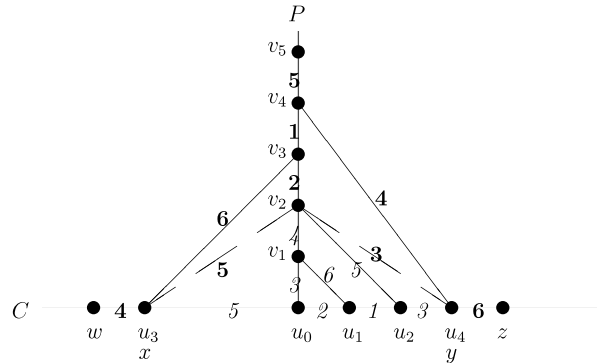
- (i)  $f(x_1y_1) = f(x_3y_3)$  and  $f(x_2y_2) = f(x_4y_4)$ , and
- (ii)  $f(y_3y_4) = f(x_1x_2)$  whenever  $y_3$  is adjacent to  $y_4$ .

**Proof.** Part (i) follows from that for each  $i$  the edges on the 4-cycle  $(x_1, x_2, x_3, x_4)$  together with the edges  $x_iy_i$  and  $x_{i+1}y_{i+1}$  use all the 6 colors, where  $x_5y_5 = x_1y_1$ .

Part (ii) follows from that the edges on the 4-cycle  $(x_3, y_3, y_4, x_4)$  together with the two edges  $x_1x_4, x_2x_3$  use all the 6 colors. See Fig. 3 for an illustration.  $\square$

We now consider the cutting tail operation for the characteristic tree of a cubic Halin graph  $G = T \cup C$ . We shall study the conditions for which such an operation preserves the fact that  $\chi'_s(G) = 6$ .

**Theorem 4.** Suppose  $P: v_1, v_2, v_3, v_4$  is a 4-tail of the characteristic tree  $T$  of a cubic Halin graph  $G = T \cup C$ , where  $L_1 = \{u_0, u_1\}$  and  $L_i = \{u_i\}$  for  $i \geq 2$ . If  $u_2$  and  $u_3$  are on the same side of  $L_1$ , then  $\chi'_s(G) = 6$  if and only if  $\chi'_s(G \ominus P_2) = 6$ .

Fig. 4. A cutting  $G \ominus P_2$ .Fig. 5. A cutting  $G \ominus P_2$ .

**Proof.** ( $\Rightarrow$ ) Suppose  $\chi'_s(G) = 6$ . Let  $f$  be a strong 6-edge-coloring of  $G$ . Without loss of generality, we may assume that  $f(xu_0) = 1, f(u_0u_1) = 2, f(u_1u_2) = 3, f(v_1u_0) = 4, f(v_1u_1) = 5$ , and  $f(v_1v_2) = 6$  as the bold faced numbers in Fig. 4. It is then the case that  $f(v_2u_2) = 1$ . Repeatedly applying Lemma 3, we have  $f(u_2u_3) = 4, f(v_2v_3) = 2, f(v_3u_3) = 5, f(u_3z) = 6$  and  $f(v_3v_4) = 3$  (see Fig. 4). In  $G \ominus P_2$ , we use the old color for edges in  $G$ , and color the new edges  $xv_2$  and  $v_2y$  by 1 and 4, respectively. It is easy to check that the new coloring is a strong 6-edge-coloring for  $G \ominus P_2$ . Hence,  $\chi'_s(G \ominus P_2) = 6$ .

( $\Leftarrow$ ) Suppose  $\chi'_s(G \ominus P_2) = 6$ . Let  $f'$  be a strong 6-edge-coloring of  $G \ominus P_2$ . Without loss of generality, assume that the colors are as in Fig. 4. We may delete the edges  $xv_2$  and  $v_2y$ , and extend the coloring to  $G$  using the colors as in Fig. 4. This gives a strong 6-edge-coloring of  $G$ , so  $\chi'_s(G) = 6$ .  $\square$

**Corollary 5.** Suppose  $n_1 + n_2 + \cdots + n_r \geq 2$ . Then  $\chi'_s(G_{n_1, n_2, \dots, n_r}) = 6$  if and only if  $\chi'_s(G_{n_1, n_2, \dots, n_r+2}) = 6$ .

**Theorem 6.** Suppose  $P: v_1, v_2, v_3, v_4, v_5$  is a 5-tail of the characteristic tree  $T$  of a cubic Halin graph  $G = T \cup C$ , where  $L_1 = \{u_0, u_1\}$  and  $L_i = \{u_i\}$  for  $i \geq 2$ . Assume  $u_2$  and  $u_3$  are on different sides of  $L_1$ , while  $u_2$  and  $u_4$  are on the same side of  $L_1$ . If  $\chi'_s(G \ominus P_2) = 6$ , then  $\chi'_s(G) = 6$ .

**Proof.** Let  $f'$  be a strong 6-edge-coloring of  $G \ominus P_2$ . By Lemma 3, without loss of generality, we may assume that  $f'(v_3v_4) = 1, f'(v_2v_3) = 2, f'(v_2y) = 3, f'(v_4u_4) = f'(wx) = 4, f'(v_4v_5) = f'(xv_2) = 5$  and  $f'(v_3u_3) = f'(u_4z) = 6$ , as the bold faced numbers shown in Fig. 5. We delete the edges  $xv_2$  and  $v_2y$  from  $G \ominus P_2$ , and extend the coloring to  $G$  using the colors shown in Fig. 5. This gives a strong 6-edge-coloring of  $G$ , so  $\chi'_s(G) = 6$ .  $\square$

Remark that unlike Theorem 4 the converse of Theorem 6 is not true. This can be seen by the example given below that  $\chi'_s(G_{1*8}) = \chi'_s(G_{1*6,2}) = 6$ , while  $\chi'_s(G_{1*6}) = 7$ . See Corollary 11.

**Corollary 7.** If  $n_1 + n_2 + \cdots + n_r \geq 2$  and  $\chi'_s(G_{n_1, n_2, \dots, n_r, 1}) = 6$ , then  $\chi'_s(G_{n_1, n_2, \dots, n_r, 1, 2}) = 6$ .

**Corollary 8.** Assume  $\chi'_s(G_{n_1, n_2, \dots, n_{r-1}, 1}) = 6$  where  $n_1 + n_2 + \cdots + n_{r-1} \geq 2$ . Then  $\chi'_s(G_{n_1, n_2, \dots, n_{r-1}, F_1, F_2, \dots, F_k, 1}) = 6$ , where each  $F_i$  is either a single positive even integer, or a list of two integers,  $(1, t)$ , for some odd integer  $t$ . In particular,  $\chi'_s(G_{n_1, n_2, \dots, n_{r-1}, 1 \star m}) = 6$  for odd  $m$ , where  $1 \star m$  stands for a sequence of  $m$  1's.

**Proof.** Assume  $\chi'_s(G_{n_1, n_2, \dots, n_{r-1}, 1}) = 6$ . It suffices to prove the result for the case  $k = 1$ . Assume  $F_1$  is a single even integer,  $F_1 = n_r$ . By Corollary 5 and Observation 2,

$$6 = \chi'_s(G_{n_1, n_2, \dots, n_{r-1}, 1}) = \chi'_s(G_{n_1, n_2, \dots, n_{r-1}, 1+n_r}) = \chi'_s(G_{n_1, n_2, \dots, n_{r-1}, n_r, 1}).$$

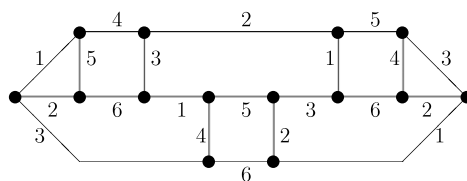


Fig. 6. A strong 6-edge-coloring for  $G_{2,2,2} = G_{2,2,1,1}$ .

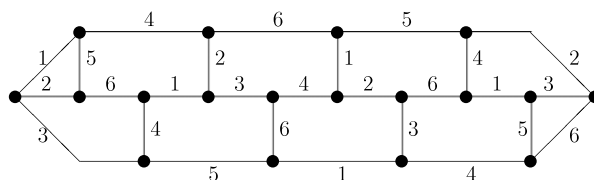


Fig. 7. A strong 6-edge-coloring for  $G_{1*8}$ .

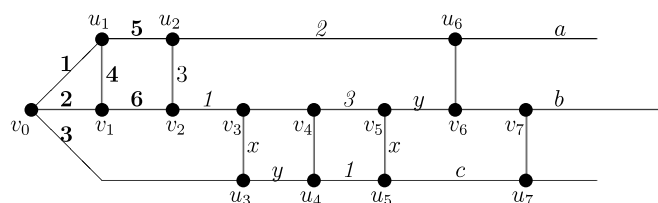


Fig. 8. In  $G_{2,3,1,n_4,n_5,...,n_r}$ , labels  $a$  and  $b$  are forced to be 4.

Next, assume that  $F_1$  is a list of two integers  $(1, t)$  for some odd  $t = 2s + 1$ . By Corollaries 5 and 7, and Observation 2,

$$6 = \chi'_s(G_{n_1, n_2, \dots, n_{r-1}, 1, 2}) = \chi'_s(G_{n_1, n_2, \dots, n_{r-1}, 1, 2(s+1)}) = \chi'_s(G_{n_1, n_2, \dots, n_{r-1}, 1, t, 1}). \quad \square$$

Let  $k$  be an even integer. Although it is known [18] that  $\chi'_s(G_k) > 6$ , there exist graphs  $G \in \mathcal{G}_k$  with  $\chi'_s(G) = 6$ . Figs. 6 and 7 show two examples. For positive integers  $x$  and  $n$ , we denote  $x \star n$  as an  $n$ -term repeated sequence of  $x$ .

By the results we have shown, one can verify that for every positive integer  $k \neq 2, 4$ , there exists  $G \in \mathcal{G}_k$  (other than necklaces) with  $\chi'_s(G) = 6$ . This is because  $\chi'_s(G_1) = \chi'_s(G_{1,1,1}) = \chi'_s(G_{2,2,1,1}) = \chi'_s(G_{1*8}) = 6$ , by Corollary 8, one gets  $\chi'_s(G_{2,2,1*m}) = 6$  for even  $m \geq 4$ , and  $\chi'_s(G_{1*n}) = 6$  for  $n \neq 2, 4, 6$ .

### 3. Cubic Halin graphs $G$ with $\chi'_s(G) = 7$

We present some cubic Halin graphs with strong chromatic index 7. In particular, we prove that for any  $k \geq 7$ , there exists  $G \in \mathcal{G}_k$  with  $\chi'_s(G) = 7$ .

Let us start with an example,  $\chi'_s(G_{2,2}) = 7$ . Suppose to the contrary that  $\chi'_s(G_{2,2}) = 6$ . Choose a strong 6-edge-coloring  $f$  for  $G_{2,2}$ . By Lemma 3(i),  $f(u_5u_7) = f(v_4v_5)$ . Since the color  $f(v_0u_3)$  has to be used by the 4-cycle  $(u_1, u_2, v_2, v_1)$ , it is the case that  $f(v_0u_3) = f(u_2v_2)$ , and so  $f(v_4v_5) = f(u_2v_2)$ , a contradiction.

**Lemma 9.** Let  $G = G_{2,3,1,n_4,n_5,...,n_r}$  with  $r \geq 4$ . If  $f$  is a strong 6-edge-coloring for  $G$ , then  $f(u_1v_1) = f(u_6u_{7+n_4}) = f(v_7v_8)$ .

**Proof.** Without loss of generality, assume that  $f(v_0u_1) = 1, f(v_0v_1) = 2, f(v_0u_3) = 3, f(v_1u_1) = 4, f(u_1u_2) = 5$ , and  $f(v_1v_2) = 6$ . See the bold faced numbers in Fig. 8. Then  $f(v_2u_2) = 3$ . By Lemma 3(i),  $f(u_2u_6) = 2, f(v_2v_3) = f(u_4u_5) = 1$ , and  $f(v_4v_5) = 3$ . See the italic numbers in Fig. 8.

Suppose  $f(v_3u_3) = x$  and  $f(u_3u_4) = y$ . Then  $x \in \{4, 5\}$  and  $y \in \{4, 5, 6\}$ . By Lemma 3,  $f(v_5u_5) = x$  and  $f(v_5v_6) = y$ . Let  $z$  be the only label in  $\{4, 5, 6\} - \{x, y\}$ . Then  $\{f(v_6v_7), f(v_6u_6)\} = \{1, z\}$ , since  $v_6v_7$  and  $v_6u_6$  cannot be labeled by 2, 3,  $x$ ,  $y$ . Let  $f(u_6u_{7+n_4}) = a$ . Then  $a \notin \{1, 2, 3, 5, y, z\}$ . Hence, it must be the case that  $a = x \neq 5$ , implying  $a = x = 4$ .

By Lemma 3(i),  $f(u_5u_7) = f(v_3v_4) = c \in \{5, 2\}$ . If  $c = 5$ , then  $f(v_6v_7) = 1$  and  $f(v_6u_6) = z = c = 5$ , which is impossible as  $f(u_1u_2) = 5$ . Hence,  $c = 2$ . Then  $f(v_7u_7) \notin \{1, 2, x, y, z\}$ , so  $f(v_7u_7) = 3$ . Consequently,  $f(v_7v_8) = b \notin \{1, 2, 3, y, z\}$ . Therefore,  $b = x = 4$ . This completes the proof.  $\square$

**Theorem 10.** The following graphs have strong chromatic index 7:

- (a)  $G_{2,3,1,n_4}$ .
- (b)  $G_{2,3,1,1,n_5,n_6,...,n_r}$  with  $r \geq 5$ .
- (c)  $G_{2,3,1,1,3,n_5}$ .

- (d)  $G_{2,3,1,3,2,n_6,n_7,\dots,n_r}$  with  $r \geq 6$ .
- (e)  $G_{2,3,1,3,4,n_6}$ .
- (f)  $G_{2,3,1,3,4,2,n_7,n_8,\dots,n_r}$  with  $r \geq 7$ .

**Proof.** For each case in the following, we suppose to the contrary that the given graph has strong chromatic index 6. Let  $f$  be a strong 6-edge-coloring for  $G$ . We shall derive a contradiction for each case.

- (a) By Corollary 5, we may assume that  $n_4 \leq 2$ . By Lemma 9,  $f(u_6u_{7+n_4}) = f(v_7v_8)$ , which contradicts the fact that the edges  $u_6u_{7+n_4}$  and  $v_7v_8$  are within distance two apart.
- (b) Since  $n_4 = 1$ , by Lemma 9,  $f(u_6u_8) = f(v_7v_8)$ , which contradicts the fact that the edges  $u_6u_8$  and  $v_7v_8$  are distance two apart.
- (c) By Corollary 5, we may assume that  $n_5 \leq 2$ . By Lemma 9 and Corollary 5,  $f(u_6u_{10}) = f(v_7v_8) = f(u_9u_{10+n_5})$ . For the case  $n_5 = 1$ , this is a contradiction as  $u_6u_{10}$  and  $u_9v_{11}$  are of distance two apart. For the case  $n_5 = 2$ , by Lemma 3(i),  $f(u_6u_{10}) = f(v_{11}v_{12})$ , and so  $f(v_{11}v_{12}) = f(u_9v_{12})$ , a contradiction.

The proofs for (d)–(f) are similar. We leave the details to the reader.  $\square$

An immediate consequence of Theorem 10 is that for every integer  $k \geq 7$ , there exists  $G \in \mathcal{G}_k$  with  $\chi'_s(G) = 7$ . This gives infinite counter examples to Conjecture 3.

#### 4. Special families

We apply the results and methods established in the previous sections to completely determine the value of  $\chi'_s(G)$  for several families of graphs  $G$  in  $\mathcal{G}_k$ .

**Corollary 11.** For  $m \geq 1$ , we have

$$\chi'_s(G_{1 \star m}) = \begin{cases} 9, & m = 2; \\ 7, & m = 4, 6; \\ 6, & \text{otherwise.} \end{cases}$$

**Proof.** For  $m = 2$ , as  $G_{1,1} = G_2$  and  $\chi'_s(G_2) = 9$  [18], so the result holds. For  $m = 4$ ,  $G_{1 \star 4} = G_{2,2}$  so  $\chi'_s(G_{1 \star 4}) = 7$ .

Because  $G_3 = G_{1 \star 3}$  and  $\chi'_s(G_3) = 6$ , so  $\chi'_s(G_{1 \star 3}) = 6$ . By Corollary 8 (letting  $n_1 = n_2 = \dots = n_{r-1} = 1$ ) and Fig. 7 the result holds for  $m = 5$  and  $m \geq 7$ .

It remains to show that  $\chi'_s(G_{1 \star 6}) > 6$ . Assume to the contrary that  $\chi'_s(G_{1 \star 6}) = 6$ . As  $G_{1 \star 6} = G_{2,1,1,2}$ , we may let  $f$  be a strong 6-edge-coloring for  $G_{2,1,1,2}$ . Without loss of generality, assume  $f(v_0u_1) = 1, f(v_0v_1) = 2, f(v_0u_3) = 3, f(u_1u_2) = 4, f(v_1u_1) = 5$ , and  $f(v_1v_2) = 6$ . Then  $f(v_2u_2) = 3, f(v_2v_3) = 1$ , and  $f(u_2u_4) = 2$ . These imply that  $\{f(v_3v_4) = f(v_3u_3)\} = \{4, 5\}$ , so  $f(u_4u_4) = f(v_0u_1) = 6$ . By Lemma 3, it must be  $f(v_6v_7) = 6$ , which is a contradiction as  $v_6v_7$  and  $v_4u_4$  are distance two apart.  $\square$

**Corollary 12.** For  $m \geq 2$ , we have

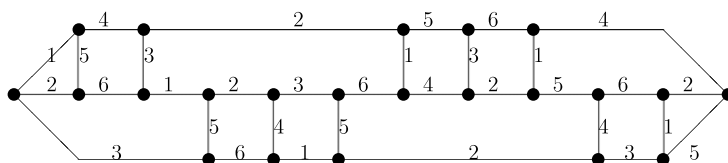
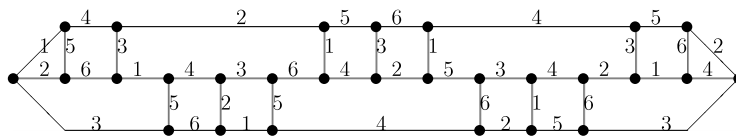
$$\chi'_s(G_{2 \star m}) = \begin{cases} 7, & m = 2; \\ 6, & \text{otherwise.} \end{cases}$$

**Proof.** At the beginning of Section 3, we have learned that  $\chi'_s(G_{2,2}) = 7$ . Fig. 6 shows a strong 6-edge coloring  $f$  for  $G_{3 \star 2}$ . In the following, we define a recursive strong 6-edge coloring for  $G_{2 \star m}$ ,  $m \geq 3$ .

Initially, let the coloring in Fig. 6 be  $f_2$ . Suppose we have a strong 6-edge coloring  $f_m$  for  $G_{2 \star m}$ . Extend  $f_m$  to a strong 6-coloring  $f_{m+1}$  for  $G_{2 \star (m+1)}$  by

$$\begin{aligned} f_{m+1}(ww') &= f_m(ww') \quad \text{if } ww' \in E(G_{2 \star m}); \\ f_{m+1}(v_{2m+1}v_{2m+2}) &= f_m(u_{2m-1}u_{2m}); \\ f_{m+1}(u_{2m-2}u_{2m+1}) &= f_{m+1}(v_{2m+2}v_{2m+3}) = f_m(v_{2m+1}u_{2m-2}); \\ f_{m+1}(u_{2m}u_{2m+3}) &= f_{m+1}(v_{2m+1}u_{2m+1}) = f_m(v_{2m+1}u_{2m}); \\ f_{m+1}(v_{2m+2}u_{2m+2}) &= f_m(v_{2m-1}v_{2m}); \\ f_{m+1}(u_{2m+1}u_{2m+2}) &= f_m(v_{2m}u_{2m}); \quad \text{and} \\ f_{m+1}(v_{2m+3}u_{2m+2}) &= f_m(v_{2m}v_{2m+1}). \end{aligned}$$

It is easy to check that the above is a strong 6-edge coloring for  $G_{2 \star (m+1)}$ . We shall leave the details to the reader.  $\square$

Fig. 9. A strong 6-edge-coloring for  $G_{2,3,3,2}$ .Fig. 10. A strong 6-edge-coloring for  $G_{2,3,3,3,2}$ .

**Corollary 13.** For  $m \geq 1$ , we have

$$\chi'_s(G_{3 \star m}) = \begin{cases} 7, & m = 2, 4, 6; \\ 6, & \text{otherwise.} \end{cases}$$

**Proof.** We first consider  $m \neq 2, 4, 5$ . Since  $\chi'_s(G_3) = \chi'_s(G_5) = 6$ , by [Observations 1](#) and [2](#) we have  $\chi'_s(G_{1,3,1}) = 6$ . By [Corollary 5](#), we get  $\chi'_s(G_{3,3,3}) = 6$ . Hence, the result holds for  $m = 1, 3$ .

Assume that  $m \geq 6$ . If  $\chi'_s(G_{2,3 \star (m-4),2}) = 6$ , then by [Corollary 5](#),  $\chi'_s(G_{4,3 \star (m-4),4}) = \chi'_s(G_{1,3 \star (m-2),1}) = \chi'_s(G_{3 \star m}) = 6$ . Hence, it is enough to find a strong 6-edge-coloring  $f$  for  $G_{2,3 \star (m-4),2}$ .

In the following we let  $f(v_0u_1) = 1, f(v_0v_1) = 2, f(v_0u_3) = 3, f(u_1u_2) = 4, f(v_1u_1) = 5$ , and  $f(v_1v_2) = 6$ . Consequently, by [Lemma 3](#),  $f(v_2v_3) = f(u_4u_5) = 1, f(u_2u_6) = f(v_7v_8) = 2$ , and  $f(u_2v_2) = f(v_4v_5) = 3$ . Since  $f(v_5v_6), f(u_2u_6) \neq 4$ , so the color 3 has to be used in the 4-cycle  $(u_6u_7v_7v_6)$ , it must be the case that  $f(v_7u_7) = 3$ .

Assume that  $m$  is even. Let  $m - 4 = 2k$ . Define  $f(v_4u_4) = 4, f(u_3u_4) = 6$ , and the remaining by the following recursive process for  $1 \leq t \leq 2k$ :

$$\begin{aligned} f(v_{3t}v_{3t+1}) &= \begin{cases} f(v_{3t-2}u_{3t-2}) & \text{if } t \text{ is even;} \\ f(v_{3t-3}v_{3t-2}) & \text{if } t \text{ is odd.} \end{cases} \\ f(v_{3t}u_{3t}) &= \begin{cases} f(u_{3t-2}u_{3t-1}) & \text{if } t \text{ is even;} \\ f(v_{3t-2}u_{3t-2}) & \text{if } t \text{ is odd.} \end{cases} \\ f(u_{3t}u_{3t+1}) &= \begin{cases} f(v_{3t-1}u_{3t-1}) & \text{if } t \text{ is even;} \\ f(u_{3t-2}u_{3t-1}) & \text{if } t \text{ is odd, } t \geq 3. \end{cases} \\ f(v_{3t+1}u_{3t+1}) &= \begin{cases} f(v_{3t-2}v_{3t-1}) & \text{if } t \text{ is even;} \\ f(v_{3t-1}u_{3t-1}) & \text{if } t \text{ is odd, } t \geq 3. \end{cases} \end{aligned}$$

By [Lemma 3](#), the colors for the remaining edges are fixed. It is not hard to see that  $f$  is a strong 6-edge-coloring for  $G_{2,3 \star (2k),2}$ . See [Fig. 9](#) for an example.

Assume  $m$  is odd. Let  $m - 4 = 2k + 1$ . Let  $f(v_3v_4) = 4, f(v_3u_3) = 5, f(u_3u_4) = 6$ , and  $f(u_4u_4) = 2$ . For  $2 \leq t \leq 2k$ , define  $f$  by the following recursive process:

$$\begin{aligned} f(v_{3t}v_{3t+1}) &= \begin{cases} f(v_{3t-3}v_{3t-2}) & \text{if } t \text{ is even;} \\ f(u_{3t-2}u_{3t-1}) & \text{if } t \text{ is odd.} \end{cases} \\ f(v_{3t}u_{3t}) &= \begin{cases} f(u_{3t-2}u_{3t-1}) & \text{if } t \text{ is even;} \\ f(v_{3t-2}u_{3t-2}) & \text{if } t \text{ is odd.} \end{cases} \\ f(u_{3t}u_{3t+1}) &= \begin{cases} f(v_{3t-1}u_{3t-1}) & \text{if } t \text{ is even;} \\ f(v_{3t-2}v_{3t-1}) & \text{if } t \text{ is odd.} \end{cases} \\ f(v_{3t+1}u_{3t+1}) &= \begin{cases} f(u_{3t-2}v_{3t-2}) & \text{if } t \text{ is even, } t \neq 2; \\ f(u_{3t-1}v_{3t-1}) & \text{if } t \text{ is odd.} \end{cases} \end{aligned}$$

Note, for  $t = 2$  in the last case above,  $f(v_7u_7) = 3$  is fixed as discussed at the beginning of the proof.

For  $t = 2k + 1$ , let  $f(v_{6k+3}v_{6k+4}) = f(u_{6k+1}v_{6k+1}), f(v_{6k+3}u_{6k+3}) = f(u_{6k+1}u_{6k+2}), f(u_{6k+3}u_{6k+4}) = f(v_{6k+1}v_{6k+2})$ , and  $f(v_{6k+4}u_{6k+4}) = f(u_{6k+2}v_{6k+2})$ .

Again, by [Lemma 3](#), the colors for the remaining edges are fixed. It is not hard to see that  $f$  is a strong 6-edge-coloring for  $G_{2,3 \star (2k+1),2}$ . See [Fig. 10](#) for an example.

Now consider  $m = 2$ . Because  $\chi'_s(G_{3,1}) = \chi'_s(G_4) > 6$ , Corollary 5 implies that  $\chi'_s(G_{3,3}) > 6$ , so  $\chi'_s(G_{3,3}) = 7$ .

For  $m = 4$ , since  $\chi'_s(G_{2,2}) > 6$ , by Corollary 5, we get  $\chi'_s(G_{4,4}) = \chi'_s(G_{1,3,3,1}) > 6$ . Use Corollary 5 twice again, we obtain  $\chi'_s(G_{3,3,3,3}) > 6$ , so  $\chi'_s(G_{3,3,3,3}) = 7$ .

For  $m = 5$ , by Theorem 10(a),  $\chi'_s(G_{2,3,1,1}) = \chi'_s(G_{2,3,2}) = 7$ . This implies, by Corollary 5,  $\chi'_s(G_{4,3,4}) = \chi'_s(G_{1,3,3,3,1}) = \chi'_s(G_{3*5}) = 7$ .  $\square$

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